

# Behavior of local cohomology modules under polarization

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**Abstract:** Let  $S = k[x_1, \dots, x_n]$  be a polynomial ring over a field  $k$  with  $n$  variables  $x_1, \dots, x_n$ ,  $\mathfrak{m}$  the irrelevant maximal ideal of  $S$ ,  $I$  a monomial ideal in  $S$  and  $I'$  the polarization of  $I$  in the polynomial ring  $S'$  with  $\rho$  variables. We show that each graded piece  $H_{\mathfrak{m}}^i(S/I)_{\mathbf{a}}$ ,  $\mathbf{a} \in \mathbf{Z}^n$ , of the local cohomology module  $H_{\mathfrak{m}}^i(S/I)$  is isomorphic to a specific graded piece  $H_{\mathfrak{m}'}^{i+\rho-n}(S'/I')_{\boldsymbol{\alpha}}$ ,  $\boldsymbol{\alpha} \in \mathbf{Z}^{\rho}$ , of the local cohomology module  $H_{\mathfrak{m}'}^{i+\rho-n}(S'/I')$ , where  $\mathfrak{m}'$  is the irrelevant maximal ideal of  $S'$ .

**Key words:** local cohomology, monomial ideal, polarization, Hochster's formula

## 1 Introduction

Let  $S = k[x_1, \dots, x_n]$  be a polynomial ring over a field  $k$  with  $n$  variables  $x_1, \dots, x_n$  and  $\mathfrak{m}$  the irrelevant maximal ideal of  $S$ . For a monomial ideal  $I$  in  $S$ , the local cohomology modules  $H_{\mathfrak{m}}^i(S/I)$  have  $\mathbf{Z}^n$ -graded structure.

Hochster described each graded piece of  $H_{\mathfrak{m}}^i(S/I)$  by using the reduced cohomology group of a simplicial complex related to  $I$  when  $I$  is square-free (see [Sta, II 4.1 Theorem]). Takayama [Tak, Theorem 1] generalized this result to the case where  $I$  is not necessarily square-free.

On the other hand, there is a technique which associates a not necessarily square-free monomial ideal  $I$  with a square-free monomial ideal, called the polarization of  $I$ , sharing many ring theoretical properties with  $I$ .

In this note, we show that the each graded piece of the local cohomology modules  $H_{\mathfrak{m}}^i(S/I)$  is isomorphic to a specific graded piece of the local cohomology module of the polarization of  $I$ .

## 2 Preliminaries

Let  $S = k[x_1, \dots, x_n]$  be a polynomial ring over a field  $k$  with  $n$  variables  $x_1, \dots, x_n$  and  $\mathfrak{m}$  the irrelevant maximal ideal of  $S$  and  $I$  a monomial ideal of  $S$ .

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For a monomial  $m = x_1^{b_1} \cdots x_n^{b_n}$  in  $S$ , we set  $\nu_i(m) = b_i$ . We denote by  $G(I)$  the minimal set of monomial generators of  $I$ . Set  $\rho_i = \max\{\nu_i(m) \mid m \in G(I)\}$  for  $i = 1, 2, \dots, n$  and  $\rho = \rho_1 + \cdots + \rho_n$ . Then the polarization of  $I$  is defined as follows.

Let  $S' = k[x_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq \rho_i]$  be the polynomial ring with  $\rho$  variables  $\{x_{ij}\}$ . For a monomial  $m$  in  $S$ , we set  $m' = \prod_{i=1}^n \prod_{j=1}^{\nu_i(m)} x_{ij}$ . Then the polarization  $I'$  of  $I$  is defined by  $I' = (m' \mid m \in G(I))S'$ . It is clear from the definition that  $I'$  is a square-free monomial ideal. Furthermore, it is known that  $\{x_{ij} - x_{i1} \mid 1 \leq i \leq n, 2 \leq j \leq \rho_i\}$  is an  $S'/I'$ -regular sequence in any order and

$$S'/(I' + (x_{ij} - x_{i1} \mid 1 \leq i \leq n, 2 \leq j \leq \rho_i)) \simeq S/I.$$

For vectors  $\mathbf{a} = (a_1, \dots, a_n)$  and  $\mathbf{b} = (b_1, \dots, b_n)$ , we denote  $\mathbf{a} \leq \mathbf{b}$  to express that  $a_i \leq b_i$  for  $i = 1, \dots, n$ . And we define  $\text{supp}_-\mathbf{a} = \{i \mid a_i < 0\}$  and call the negative support of  $\mathbf{a}$ . We set  $\mathbf{0} = (0, 0, \dots, 0)$ ,  $\mathbf{1} = (1, 1, \dots, 1)$  and  $\boldsymbol{\rho} = (\rho_1, \rho_2, \dots, \rho_n)$ .

We denote the cardinality of a finite set  $X$  by  $|X|$  and the set  $\{1, 2, \dots, n\}$  by  $[n]$ .

Here we recall the result of Takayama.

**Theorem 2.1 ([Tak, Theorem 1])** *Let  $S$  and  $I$  be as above and  $\mathbf{a} \in \mathbf{Z}^n$ . Set  $\Delta_{\mathbf{a}} = \{F \setminus \text{supp}_-\mathbf{a} \mid [n] \supset F \supset \text{supp}_-\mathbf{a}, \forall m \in G(I) \exists i \in [n] \setminus F; a_i < \nu_i(m)\}$ . Then*

$$H_{\mathbf{m}}^i(S/I)_{\mathbf{a}} \simeq \tilde{H}^{i-|\text{supp}_-\mathbf{a}|-1}(\Delta_{\mathbf{a}}; k).$$

Note that  $\Delta_{\mathbf{a}}$  is a simplicial complex with vertex set  $[n]$ . We call  $\Delta_{\mathbf{a}}$  the Takayama complex. Note also that if  $a_i \geq \rho_i$ , then  $\Delta_{\mathbf{a}}$  is a cone over  $i$  and all the reduced cohomology modules vanish. Therefore  $H_{\mathbf{m}}^i(S/I)_{\mathbf{a}} = 0$  if  $\mathbf{a} \not\leq \boldsymbol{\rho} - \mathbf{1}$ .

### 3 Main theorem

Now we state the main result of this paper.

**Theorem 3.1** *With the notation in previous section, assume that  $\mathbf{a} \leq \boldsymbol{\rho} - \mathbf{1}$ . Set*

$$\alpha_i = \begin{cases} (\underbrace{0, \dots, 0}_{a_i+1}, \underbrace{-1, \dots, -1}_{\rho_i-a_i-1}) & \text{if } a_i \geq 0, \\ (\underbrace{-1, \dots, -1}_{\rho_i}) & \text{if } a_i < 0 \end{cases}$$

and  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbf{Z}^\rho$ . Then

$$H_{\mathfrak{m}}^i(S/I)_\alpha \simeq H_{\mathfrak{m}'}^{i+\rho-n}(S'/I')_\alpha,$$

where  $\mathfrak{m}'$  is the irrelevant maximal ideal of  $S'$ .

**proof. Step 1.** We first consider the case where  $\alpha = \rho - \mathbf{1}$ .

First note that local cohomology modules  $H_{\mathfrak{m}'}^i(S'/I')$  have not only the  $\mathbf{Z}^\rho$ -grading but also the  $\mathbf{Z}^n$ -grading by setting  $\deg x_{ij} = \mathbf{e}_i$ , where  $\mathbf{e}_i$  is the  $i$ -th fundamental vector in  $\mathbf{Z}^n$ .

We denote by  $C^\bullet$  the Čech complex with respect to  $\{x_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq \rho_i\}$  and  $K_\bullet$  the Koszul complex with respect to  $\{x_{ij} - x_{i1} \mid 1 \leq i \leq n, 2 \leq j \leq \rho_i\}$ . Set  $M^{p,q} = C^p \otimes_{S'} K_{\rho-n-q} \otimes_{S'} S'/I'$ . Then  $M^{\bullet,\bullet}$  is a third quadrant double complex which has a  $\mathbf{Z}^n$ -graded structure. Set  $\{^r E_r\}$  and  $\{'' E_r\}$  be the spectral sequences arising from  $M^{\bullet,\bullet}$ .

Since  $\{x_{ij} - x_{i1} \mid 1 \leq i \leq n, 2 \leq j \leq \rho_i\}$  is an  $S'/I'$ -regular sequence, we see that

$$^r E_1^{p,q} \simeq \begin{cases} C^p \otimes S/I & q = \rho - n, \\ 0 & \text{otherwise.} \end{cases}$$

And the horizontal complex  $^r E_1^{\bullet,\rho-n}$  is isomorphic to the Čech complex with respect to  $\underbrace{x_1, x_1, \dots, x_1}_{\rho_1}, \underbrace{x_2, x_2, \dots, x_2}_{\rho_2}, \dots, \underbrace{x_n, x_n, \dots, x_n}_{\rho_n}$ . Therefore

$$^r E_2^{p,q} \simeq \begin{cases} H_{\mathfrak{m}}^p(S/I) & q = \rho - n, \\ 0 & \text{otherwise.} \end{cases}$$

So the spectral sequence  $\{^r E_r\}$  collapses and we see that

$$H^i(\text{Tot}(M^{\bullet,\bullet})) \simeq ^r E_2^{i-\rho+n, \rho-n} \simeq H_{\mathfrak{m}}^{i-\rho+n}(S/I)$$

for any  $i \in \mathbf{Z}$ , where  $\text{Tot}(M^{\bullet,\bullet})$  is the total complex of  $M^{\bullet,\bullet}$ .

Next we consider  $\{'' E_r\}$ . It is clear that  $'' E_1^{p,q} \simeq K_{\rho-n-q} \otimes_{S'} H_{\mathfrak{m}'}^p(S'/I')$ . Since  $I'$  is square-free,  $H_{\mathfrak{m}'}^p(S'/I')_\alpha = 0$  if  $\alpha \not\leq \mathbf{0}$  by the remark after Theorem 2.1. Therefore  $H_{\mathfrak{m}'}^p(S'/I')_l = 0$  if  $l > 0$ ,  $l \in \mathbf{Z}$ , where  $H_{\mathfrak{m}'}^p(S'/I')_l$  denotes the total degree  $l$  piece of  $H_{\mathfrak{m}'}^p(S'/I')$ .

Since  $K_{\rho-n-q}$  is a free  $S'$ -module with free basis consisting of total degree  $\rho - n - q$  elements, we see that

$$('' E_1^{p,q})_{\rho-n} = 0 \quad \text{if } q \neq 0.$$

Therefore total degree  $\rho - n$  piece of  $\{'' E_r\}$  collapses and we see that

$$H^i(\text{Tot}(M^{\bullet,\bullet}))_{\rho-n} \simeq ('' E_1^{i,0})_{\rho-n} \simeq H_{\mathfrak{m}'}^i(S'/I')_0$$

for any  $i \in \mathbf{Z}$ .

So

$$H_{\mathbf{m}}^{i-\rho+n}(S/I)_{\rho-n} \simeq H_{\mathbf{m}'}^i(S'/I')_0$$

for any  $i \in \mathbf{Z}$ . By the remark after Theorem 2.1, we see that  $H_{\mathbf{m}}^j(S/I)_{\rho-n} = H_{\mathbf{m}}^j(S/I)_{\rho-1}$  and  $H_{\mathbf{m}'}^j(S'/I')_0 = H_{\mathbf{m}'}^j(S'/I')_0$  for any  $j \in \mathbf{Z}$ . This means

$$H_{\mathbf{m}}^i(S/I)_{\rho-1} \simeq H_{\mathbf{m}'}^{i+\rho-n}(S'/I')_0$$

and this is what we wanted to prove.

**Step 2.** Next we consider the case where  $a_i = \rho_i - 1$  or  $a_i < 0$  for any  $i = 1, \dots, n$ .

By changing the subscripts, we may assume that  $a_i = \rho_i - 1$  for  $i = 1, \dots, m$  and  $a_i < 0$  for  $i = m+1, \dots, n$ . Set  $S_+ = k[x_1, \dots, x_m]$ ,  $I_+ = IS[x_{m+1}^{-1}, \dots, x_n^{-1}] \cap S_+$  and  $\mathbf{m}_+$  the irrelevant maximal ideal of  $S_+$ . Note that  $I_+$  is the monomial ideal of  $S_+$  generated by the monomials obtained by substituting 1 to  $x_{m+1}, \dots, x_n$  of the monomials in  $I$ . Note also that if we denote the Takayama complex with respect to  $I_+$  and  $(a_1, \dots, a_m)$  by  $\Delta_+$ , then  $\Delta_+ = \Delta_{\mathbf{a}}$ . Therefore

$$\begin{aligned} H_{\mathbf{m}}^i(S/I)_{\mathbf{a}} &\simeq \tilde{H}^{i-|\text{supp}_- \mathbf{a}|-1}(\Delta_{\mathbf{a}}; k) \\ &\simeq \tilde{H}^{i-(n-m)-1}(\Delta_+; k) \\ &\simeq H_{\mathbf{m}_+}^{i-n+m}(S_+/I_+)_{(a_1, \dots, a_m)} \end{aligned}$$

since  $|\text{supp}_- \mathbf{a}| = n - m$  and  $|\text{supp}_-(a_1, \dots, a_m)| = 0$ .

We also set  $S'_+ = k[x_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq \rho_i]$ ,  $I'_+ = I'S'[x_{ij}^{-1} \mid m+1 \leq i \leq n, 1 \leq j \leq \rho_i] \cap S'_+$  and  $\mathbf{m}'_+$  the irrelevant maximal ideal of  $S'_+$ . Then it is easily verified that  $I'_+$  is the polarization of  $I_+$ . And we see that  $H_{\mathbf{m}'}^{i+\rho-n}(S'/I')_{\mathbf{a}} \simeq H_{\mathbf{m}'_+}^{i+\rho_1+\dots+\rho_m-n}(S'_+/I'_+)_{(\alpha_1, \dots, \alpha_m)}$  by the same argument above. Therefore, we can reduce this case to the case of step 1.

**Step 3.** Finally, we consider the general case.

Assume that  $0 \leq a_n < \rho_n - 1$ . Consider the “partial polarization with respect to  $x_n$ ”. That is, set  $S'' = S[x_{nj} \mid a_n+2 \leq j \leq \rho_n]$  and for a monomial  $m$  in  $S$ , set

$$m'' = \begin{cases} m & \text{if } \nu_n(m) \leq a_n + 1, \\ \prod_{i=1}^{n-1} x_i^{\nu_i(m)} x_n^{a_n+1} \prod_{j=a_n+2}^{\nu_n(m)} x_{nj} & \text{if } \nu_n(m) \geq a_n + 2. \end{cases}$$

Set also  $I'' = (m'' \mid m \in G(I))$ . Then the polarization of  $I''$  is the same as that of  $I$ .

And if we denote the Takayama complex with respect to  $I''$  and  $(a_1, \dots, a_{n-1}, a_n, \underbrace{-1, \dots, -1}_{\rho_n - 1 - a_n})$  by  $\Delta''$ , then we see that  $\Delta'' = \Delta_{\mathbf{a}}$ . Therefore

$$\begin{aligned} H_{\mathbf{m}}^i(S/I)_{\mathbf{a}} &\simeq \tilde{H}^{i-|\text{supp}_- \mathbf{a}|-1}(\Delta_{\mathbf{a}}; k) \\ &\simeq \tilde{H}^{i-|\text{supp}_- (a_1, \dots, a_{n-1}, a_n, -1, \dots, -1)|-1+(\rho_n-1-a_n)}(\Delta''; k) \\ &\simeq H_{\mathbf{m}''}^{i+\rho_n-(a_n+1)}(S''/I'')_{(a_1, \dots, a_{n-1}, a_n, -1, \dots, -1)}. \end{aligned}$$

So we can reduce the proof to the case where  $a_n = \rho_n - 1$ .

Using this argument repeatedly, we may assume that  $a_i \geq 0 \Rightarrow a_i = \rho_i - 1$ , i.e., we can reduce the proof of the theorem to the case of step 2. ■

**Remark 3.2** Step 1 of the proof of Theorem 3.1 can also be proved by using [Sba, Corollary 5.2], insted of the spectral sequence argument.

## References

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